Target Input Model with Learning, Derivations

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These derivations are not part of the official forthcoming version of Vasilaky and Leonard (2016) in Economic Development and Cultural Change. Rather, they are supplementary derivations, particularly for those who are unfamiliar with the target input model, and who have not derived the intermediary steps shown here. The syntax and lettering does not exactly correspond to what is used in the published version. We changed some of the notation in the final version for ease of reading.

Target Input Model

There are two main sources for the target input model. Udry and Bardhan's Development Microeconomics Handbook, as well as Foster and Rosenzweig (1995). They use different notations.

Farmer i chooses an input level (or, time to apply inputs): θ_{it}, at time t, to maximize profits. The ideal input is $\widetilde{\theta}_{it}$, and farmer's profits are larger the closer is θ_{it} to $\widetilde{\theta}_{it}$.

Profits q for farmer i in time t:

$$q_{it} = 1 - \left(\theta_{it} - \widetilde{\theta}_{it}\right)^2 \quad (1)$$

Choose: θ_{it}; Target: $\widetilde{\theta}_{it}$

$\widetilde{\theta}_{it}$ is determined by

$$\widetilde{\theta}_{it} = \theta^* + \mu_{it} \to \mu_{it} \sim N\left(0, \vartheta_{\mu}^2\right) \quad (2)$$

Note: Udry/Bardhan Book Version $\to$ $\quad (\kappa_{it} = \kappa^* + \mu_{it})$

$\theta^* \sim N\left(\theta_t^*, \vartheta_{\widetilde{\theta}_{it}}^2\right) \leftarrow$

Maximization of expected profit implies that: $\left(2\left(\theta_{it} - \widetilde{\theta}_{it}\right) = 0\right)$

$$\theta_{it} = E_t\left(\widetilde{\theta}_{it}\right) = \theta_t^* \Rightarrow$$
$$E_t\left(q_{it}\right) = 1 - \vartheta^2_{\widetilde{\theta}_{it}} - \vartheta^2_{\mu} \quad (3)$$

Expected profits increase as $\vartheta^2_{\widetilde{\theta}_{it}}$ decreases or as the individual learns about true θ^*.

(a) Derivation of $E_t(q_{it}) = 1 - \vartheta^2_{\widetilde{\theta}_{it}} - \vartheta^2_{\mu}$

$$E_t(q_{it}) = 1 - E_t(\theta^2_{it}) + 2E_t\left(\theta_{it}\widetilde{\theta}_{it}\right) - E_t\left(\widetilde{\theta}^2_{it}\right)$$

Now $\theta_{it} = E_t\left(\widetilde{\theta}_{it}\right) = \theta^*_t$ from Maxmization

$$\Rightarrow E_t(\theta^2_{it}) = E_t\left(\theta^{*2}_t\right) \quad ①$$
$$\Rightarrow \text{And } 2E_t\left(\theta_{it}\widetilde{\theta}_{it}\right) = 2E_t\left(\theta^*_t\widetilde{\theta}_{it}\right)$$
$$= 2E_t\left(\theta^*_t\left(\theta^* + \mu_{it}\right)\right) \text{ Plugging in (2)}$$
$$= 2\theta^*_t E_t(\theta^*) + 2\theta^*_t E(\mu_{it})$$
$$= 2\theta^{*2}_t + 0 = 2\theta^{*2}_t \quad ②$$
$$\text{And } E_t\left(\widetilde{\theta}^2_{it}\right) = E_t\left(\theta^* + \mu_{it}\right)^2 \text{ by (2)}$$
$$= E_t\left(\theta^{*2}\right) + 2E_t(\theta^*\mu_{it}) + E_t(\mu^2_{it})$$
$$= E_t\left(\theta^{*2}\right) + E_t(\mu^2_{it}) = E_t\left(\theta^{*2}\right) + \vartheta^2_{\mu} \quad ③$$

Putting ①②③ together:

$$E_t(q_{it}) = 1 - E_t\left(\theta^{*2}_t\right) + 2\theta^2_t - \left(E_t\left(\theta^{*2}\right) + \vartheta^2_{\mu}\right)$$
$$= 1 - \theta^{*2}_t + 2\theta^{*2}_t - E_t\left(\theta^{*2}\right) - \vartheta^2_{\mu}$$
$$= 1 - \left[E_t\left(\theta^{*2}\right) - 2E(\theta^*)E(\theta^*) + E(\theta^*)^2\right] - \vartheta^2_{\mu} = 1 - \vartheta^2_{\widetilde{\theta}_{it}} - \vartheta^2_{\mu}$$

which is Equation (3), pg 155 in Development Microeconomics By Pranab Bardhan, Christopher Udry.

Farmer i's variance of her beliefs about θ^* in period t-1 is $\vartheta^2_{\widetilde{\theta}_{i,t-1}}$, after observing $\widetilde{\theta}_{i,t}$ and applying Baye's rule for a Normal distribution with Normal prior. (b)

For observation model $\widetilde{\theta}_{it}|\theta^* \sim N\left(\theta^*, \vartheta^2_\mu\right)$ and priors on $\widetilde{\theta}_{it} \sim N\left(\theta^*_{i,t-1}, \vartheta^2_{\widetilde{\theta}_{i,t-1}}\right)$

The posterior variance of $\widetilde{\theta}$, $\vartheta^2_{\widetilde{\theta}_{it}}$ after one update is: (b) [1]

$$\vartheta^2_{\widetilde{\theta}_{it}} = \frac{\vartheta^2_{\widetilde{\theta}_{i,t-1}}\vartheta^2_\mu}{\vartheta^2_{\widetilde{\theta}_{i,t-1}} + \vartheta^2_\mu} \quad (4)$$

$$= \frac{1}{\frac{1}{\vartheta^2_{\widetilde{\theta}_{i,t-1}} + \frac{1}{\vartheta^2_\mu}}} \text{ if we plug in for (t-1)}$$

$$= \frac{1}{\frac{1}{\frac{1}{\frac{1}{\vartheta^2_{\widetilde{\theta}_{i,t-2}} + \frac{1}{\vartheta^2_\mu}}}} + \frac{1}{\vartheta^2_\mu}} = \frac{1}{\frac{1}{\vartheta^2_{\widetilde{\theta}_{i,t-2}}} + 2\frac{1}{\vartheta^2_\mu}}$$

We can see that if we continue to iterate until until period 0:

$$= \frac{1}{\frac{1}{\vartheta^2_{\widetilde{\theta}_{io}}} + N_{t-1}\vartheta^2_\mu}$$

where N_{t-1} represents the number of iterations from period t-1 to period 0,

Define precision as $P_{io} = \frac{1}{\vartheta^2_{\widetilde{\theta}_{io}}}$ and $P_\mu = \frac{1}{\vartheta^2_\mu}$,

$$\vartheta^2_{\widetilde{\theta}_{it}} = \frac{1}{P_{io} + N_{t-1}\vartheta^2_\mu} \text{ (4')}$$

$\lim_{N\to\infty} \vartheta^2_{\widetilde{\theta}_{i,t}} = 0 \Rightarrow \lim_{N\to\infty} E\left(q_i t\right) = 1 - \vartheta^2_\mu$

[1] See Derivation on next page 4.

Here we do a side derivation for what the posterior distribution with normally distributed data with normal priors, with a change of notation.

Observation model $y|\mu \sim \mathrm{N}(\mu, \vartheta^2)$, Normal prior $\sim N(m, S^2)$

We consider the a normal prior of mean m and variance s^2.

$f(\mu) = \frac{1}{\sqrt{2\pi s^2}} e^{-\frac{1}{2s^2}(\mu - m)^2}$

The posterior is proportional to the prior times the likelihood. Baye's rule $f(\mu|y) \sim f(y|\mu) f(\mu)$.

$$= e^{\left\{-\frac{(y-\mu)^2}{2\vartheta^2}\right\}} e^{\left\{-\frac{(\mu-m)^2}{2S^2}\right\}}$$

$$= e^{\left\{-\frac{(y-\mu)^2}{2\vartheta^2} - \frac{(y-m)^2}{2S^2}\right\}}$$

$$= e^{\left\{-\frac{1}{2}\left\{\frac{\mu^2 - 2\mu y + y^2}{\vartheta^2} + \frac{m^2 - 2m\mu + \mu^2}{S^2}\right\}\right\}}$$

$$= e^{-\frac{1}{2}\left\{\frac{S^2\mu^2 - 2\mu y S^2 + y^2 S^2}{\vartheta^2 S^2} + \frac{\vartheta^2 m^2 - 2m\mu\vartheta^2 + \mu^2\vartheta^2}{\vartheta^2 S^2}\right\}}$$

Drop $-\frac{1}{2}\left\{\frac{y^2 S^2 + \vartheta^2 m^2}{\vartheta^2 S^2}\right\}$ since constants (known)

$$\text{So } f(\mu|y) \sim e^{-\frac{1}{2}\left\{\frac{S^2\mu^2 - 2\mu y S^2}{\vartheta^2 S^2} + \frac{-2m\mu\vartheta^2 + \mu^2\vartheta^2}{\vartheta^2 S^2}\right\}}$$

$$= e^{-\frac{1}{2}\left\{\frac{(S^2 + \vartheta^2)\mu^2 - 2\mu(yS^2 + m\vartheta^2)}{S^2\vartheta^2}\right\}}$$

$$= e^{-\frac{1}{2}\left\{\frac{\mu^2 - 2\mu\frac{yS^2 + m\vartheta^2}{S^2 + \vartheta^2}}{\frac{S^2\vartheta^2}{S^2 + \vartheta^2}}\right\}}$$

$$= e^{-\frac{1}{2}\left\{\frac{\left(\mu - \frac{yS^2 + m\vartheta^2}{S^2 + \vartheta^2}\right)^2}{\frac{S^2\vartheta^2}{S^2 + \vartheta^2}}\right\}}$$

The 2nd term is a constant when you square this out, and can be dropped.

So $f(\mu|y)$ is normal with moments:

$$\text{mean: } \frac{yS^2 + m\vartheta^2}{S^2 + \vartheta^2} \qquad \text{sd: } \frac{S^2\vartheta^2}{S^2 + \vartheta^2}$$

We used the above posterior sd in equation (4). Now back to the target input model.

From Pg (3), if $\lim_{N\to\infty} \vartheta^2{}_{\widetilde{\theta}_{it}} = 0$, then in the limit i's beliefs about θ^* are distributed as $\sim N(\theta^*_t, 0)$.

Learning From Others

Now suppose person i can observe person j's input choice, she observes person j's choice $\widetilde{\theta}_{jt} + \varepsilon_{jt}$, or $\theta^* + \mu_{jt} + \varepsilon_{jt}$.

For the moment, we assume that information flow, and the errors μ_{it} and μ_{jt}, are independent (Cov(μ_{it}, μ_{jt})=0). Now farmer i has an additional update regarding $\widetilde{\theta}_{it}$ using her neighbors' plot. A second update takes the following form.

$$\text{Let } \vartheta^2_v = \frac{\vartheta^2_{\widetilde{\theta}_{i,t-1}} \vartheta^2{}_\mu}{\vartheta^2_{\widetilde{\theta}_{i,t-1}} + \vartheta^2_\mu} \text{ 1st update on the priors.}$$

Let $\vartheta^2_z = \vartheta^2_\mu + \vartheta^2_\epsilon$, which is our additional data coming from an update from peers.

where $\widetilde{\theta}_{jt} \sim N(\theta^*, \vartheta^2_z)$

Following our derivation on Pg 4, the 2nd update takes the following form:

$$\frac{\vartheta^2_z \vartheta^2_v}{\vartheta^2_z + \vartheta^2_v} = \frac{\vartheta^2_z \dfrac{\vartheta^2_{\widetilde{\theta}_{i,t-1}} \vartheta^2{}_\mu}{\vartheta^2_{\widetilde{\theta}_{i,t-1}} + \vartheta^2_\mu}}{\vartheta^2_z + \dfrac{\vartheta^2_{\widetilde{\theta}_{i,t-1}} \vartheta^2{}_\mu}{\vartheta^2_{\widetilde{\theta}_{i,t-1}} + \vartheta^2_\mu}} = \frac{\dfrac{\vartheta^2_z \vartheta^2_{\widetilde{\theta}_{i,t-1}} \vartheta^2_\mu}{\vartheta^2_{\widetilde{\theta}_{i,t-1}} + \vartheta^2_\mu}}{\dfrac{\vartheta^2_z \vartheta^2_{\widetilde{\theta}_{i,t-1}} + \vartheta^2_z \vartheta^2{}_\mu + \vartheta^2_{\widetilde{\theta}_{i,t-1}} \vartheta^2{}_\mu}{\vartheta^2_{\widetilde{\theta}_{i,t-1}} + \vartheta^2_\mu}}$$

$$= \frac{\vartheta^2_z \vartheta^2_{\widetilde{\theta}_{i,t-1}} \vartheta^2_\mu}{\vartheta^2_z \vartheta^2_{\widetilde{\theta}_{i,t-1}} + \vartheta^2_z \vartheta^2{}_\mu + \vartheta^2_{\widetilde{\theta}_{i,t-1}} \vartheta^2_\mu} = \frac{1}{\dfrac{1}{\vartheta^2{}_\mu} + \dfrac{1}{\vartheta^2_{\widetilde{\theta}_{i,t-1}}} + \dfrac{1}{\vartheta^2_z}} = \frac{1}{\dfrac{1}{\vartheta^2{}_\mu} + \dfrac{1}{\vartheta^2_{\widetilde{\theta}_{i,t-1}}} + \dfrac{1}{\vartheta^2{}_\mu + \vartheta^2{}_\epsilon}}$$

Let $P_v = \dfrac{1}{\vartheta^2{}_\mu + \vartheta^2{}_\epsilon}$.

Now if we substitute in for $\vartheta^2_{\widetilde{\theta}_{i,t-1}}$, the farmer will update her priors using her own observations of $\widetilde{\theta}_{it}$ and her, S, neighbors $\widetilde{\theta}_{jt}$, and the variance of her beliefs about θ^*, between time $t-1$ and time 0 are:

$$\vartheta^2_{\widetilde{\theta}_{i,t-1}} = \frac{1}{P_{io} + N_{t-1}P_{\mu} + S_{t-1}P_{v}}$$

For the remainder of the model found in Vasilaky and Leonard (2016), see the Appendix in the paper. The pre-print version can be found on Columbia University's Academic Commons.